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AN AVERAGING THEOREM FOR DISTRIBUTED CONSERVATIVE SYSTEMS AND ITS APPLICATION TO VON KARMAN'S EQUATIONS*

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An averaging theorem, of the Krylov-Bogolyubov-Mitropols'kii type, is proved for oscillatory processes in spatially-multidimensional conservative systems. Von Kármán's equations are considered as an example.

1. *Statement of the problem.* Oscillatory processes in distributed conservative systems can be described by means of Hamiltonian equations in an infinite-dimensional phase space Z equipped with a symplectic structure /1-3/. As in the finite-dimensional case /4/, writing the equations in Hamiltonian form is equivalent to expressing them as variational principles (the latter approach is much more popular for the equations of mechanics of continuous media; see, e.g., /5/).

Equipping Z with a symplectic structure is equivalent to defining a Poisson bracket $[H_1, H_2]$ for functionals $H_1, H_2 : Z \to \mathbf{R}$ (see /2-4/). In the simplest case, Z is a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and the Poisson bracket is

$$[H_1, H_2](u) = \langle \nabla H_1(u), J\nabla H_2(u)\rangle,\ u \in Z \tag{1.1}$$

Here J is an antiselfadjoint operator in Z (possibly non-bounded) and ∇ is the gradient relative to the scalar product in Z, i.e.,

$$H_1(u + \varepsilon v) = H_1(u) + \varepsilon\langle\nabla H_1(u), v\rangle + o(\varepsilon)$$

With this symplectic structure, a functional H on Z is associated with the following Hamiltonian equation:

$$u^\cdot = J\nabla H(u),\ u = u(t) \in Z \tag{1.2}$$

In most cases $Z = L_2(\Omega; \mathbf{R}^N)$, where Ω is an n-dimensional region $(n \geqslant 1)$, and H is a functional of variational calculus /1, 2, 5/. Then $\nabla H(u) = \delta H/\delta u(x)$ is the variational derivative of H. A more complicated example of an infinite-dimensional symplectic space is presented below (Sect.3), in connection with von Kármán's equations.

We will consider the problem of small oscillations in system (1.2). To that end we focus our attention on the quadratic term in H and substitute $u = \varepsilon z$. This gives an equation for $z(t)$ with Hamiltonian $H_0(z) = \langle Az, z\rangle/2 + \varepsilon H_\Delta(z, \varepsilon)$, where A is a selfadjoint operator:

$$z^\cdot = J(Az + \varepsilon\nabla H_\Delta(z, \varepsilon)) \tag{1.3}$$

We shall assume that the spectrum of the operator JA is pure imaginary.

An averaged m-th approximation solution $(m \geqslant 0)$ of Eq.(1.3) is defined as a curve $z_*(t)$ which for $0 \leqslant t \leqslant L(\varepsilon)$, where $\varepsilon L(\varepsilon) \to \infty$ as $\varepsilon \to 0$, differs from the exact solution $z(t)$ by $o(\varepsilon^m)$ /6/.

The averaging problem for equations of type (1.3) describing oscillations of spatially one-dimensional systems has been intensively researched (see, e.g., /7, 8/ and the bibliography therein); Maslov and his students have averaged the solutions of equations of type (1.2) with rapidly oscillating initial conditions /9, 10/.

Our purpose in this paper is to average Eqs.(1.3) without assuming that the system is spatially one-dimensional. More precisely, we wish to construct averaged trajectories of (1.3) corresponding to non-resonant conditionally periodic solutions of the unperturbed linear equation

$$z^\cdot = JAz \tag{1.4}$$

(i.e., solutions of Eq.(1.4) under which a finite number of modes are excited). Solutions of Eq.(1.3) that are close to conditionally periodic solutions of (1.4) must also be found when one is studying oscillations in non-autonomous Hamiltonian systems of the form

$$z^\cdot = JAz + \varepsilon J\nabla H_\Delta(z, \omega_1 t, \ldots, \omega_n t) \tag{1.5}$$

$$\omega_j \in \mathbf{R},\ j = 1, \ldots, n$$

where the Hamiltonian $H_\Delta(z, \zeta_1, \ldots, \zeta_n)$ is 2π-periodic in $\zeta_1, \ldots, \zeta_n$.

In fact, we define auxiliary cyclic variables $q_1, \ldots, q_n$, $q = (q_1, \ldots, q_n) \in T^n = \mathbf{R}^n/(2\pi\mathbf{Z}^n)$, and variables $(I_1, \ldots, I_n) = I \in \mathbf{R}^n$. System (1.5) is equivalent to the autonomous Hamiltonian

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system with Hamiltonian $I\cdot\omega+\langle Az, z\rangle/2+\varepsilon H_\Delta(z, q)$ in the extended phase space $T^n\times\mathbf{R}^n\times\mathbf{Z}$:

$$q_j^{\cdot}=\omega_j,\ I_j^{\cdot}=-\varepsilon\frac{\partial}{\partial q_j}H_\Delta(z, q),\ z^{\cdot}=J(Az+\varepsilon\nabla_z H(z, q)) \tag{1.6}$$

System (1.6) in turn is equivalent to a certain autonomous system of type (1.3) in the space $\mathbf{R}^{2n}\times Z$, considered in the neighbourhood of a conditionally periodic solution of the linear system (see Eq.(4.1) below).

Let us assume that Z is spanned by an orthonormal basis $\{\varphi_j^\pm \mid j=1, 2, \ldots\}$ with the following properties:

A) $J\varphi_j^\pm=\mp\lambda_{jJ}\varphi_j^\mp,\quad A\varphi_j^\pm=\lambda_{jA}\varphi_j^\mp,\quad \forall j\geqslant 1,$

B) $\lambda_{jA}=K_A j^{d_A}+o(j^{d_A}),\quad \lambda_{jJ}=K_J j^{d_J}+o(j^{d_J}),\quad d_A, d_J\geqslant 0$.

In particular, the operator JA has a pure imaginary spectrum $\{\pm i\lambda_j \mid j\geqslant 1\}$, where $\lambda_l=\lambda_{lA}\lambda_{lJ}=K_0 j^{d_1}+o(j^{d_1})$, $d_1=d_A+d_J$. Hence all solutions of Eq.(1.4) are almost periodic functions of time, and the solutions corresponding to excitation of only a finite number of modes are conditionally periodic.

For example, solutions in which the first n modes are excited are

$$z(t)=\sum_{k=1}^n\sqrt{2I_k}(\cos(\lambda_k t+\nu_k)\varphi_k^+ +\sin(\lambda_k t+\nu_k)\varphi_k^-) \tag{1.7}$$
$$\nu_k\in[0, 2\pi),\quad \lambda_k=\lambda_{kA}\lambda_{kJ},\quad I_k>0,\quad k=1,\ldots,n$$

The trajectory (1.7) lies on an n-dimensional torus

$$T^n(I)=\{y_1^+\varphi_1^+ + y_1^-\varphi_1^- +\ldots+y_n^-\varphi_n^- \mid y_j^{+2}+y_j^{-2}=2I_j, \forall j\}\subset Z$$

It turns out (see Sect.2, Theorem 1) that under certain conditions implying lack of resonance, if the initial condition u_0 of Eq.(1.3) is distant from $T^n(I)$ by a quantity of the order of ε^a, $0<a\leqslant 1$, then the averaged trajectory of order $m, 0\leqslant m<a$, is a curve of type (1.7). The frequency vector $(\lambda_1,\ldots,\lambda_n)$ characterizing the curve is replaced by a similar vector $\omega^1\in\mathbf{R}^n$, which is determined by averaging over $T^n(I)$ certain quantities derived from the perturbation εH_Δ.

In Sect.3, as an illustration of system (1.3), we shall consider von Kármán's equations for the small oscillations of a thin plate (/11/, Chap.1, Sect.4; /12/):

$$u_1^{\cdot\cdot}+a_1\Delta^2 u_1-\sqrt{\varepsilon}[u_1, u_2]'=0,\ a_2\Delta^2 u_2+\sqrt{\varepsilon}[u_1, u_1]'=0 \tag{1.8}$$
$$a_1, a_2>0,\ u_1=u_1(t, x),\ u_2=u_2(t, x),\ x=(x, x_2)$$
$$[u, v]'=D_1^2uD_2^2v+D_2^2uD_1^2v-2D_1D_2uD_1D_2v,\ D_i=\partial/\partial x_i$$

Here $\Delta^2=\Delta\Delta$ is the iterated Laplacian (with respect to the variables x).

System (1.8) is reducible to the form (1.3), and it will follow from Theorem 1 that if the initial conditions $u_1(0), u_1^{\cdot}(0)$ can be approximated to within $\varepsilon^a, 0<a\leqslant 1$, by sums of n eigenfunctions of the operator Δ^2, then there exists one and only one solution for $0\leqslant t\leqslant L(\varepsilon)$, where $L(\varepsilon)\gg\varepsilon^{-1}$. The effect of the non-linear increment to the solution "in the large" is to modify the natural frequencies of the linear system by quantities of the order of ε, while the eigenfunctions themselves remain unchanged, provided that the initial set of frequencies is non-resonant.

2. *Statement of the theorem.* In the sequel $C, C_1, C_2, \ldots$ will denote various positive constants independent of ε, and $U_a(B)$ will denote the open sphere of radius $a>0$ centred at the zero of a Hilbert space B.

Let Y be the closed linear span in Z of the vectors $\{\varphi_k^\pm \mid k\geqslant n+1\}$. For $z=\sum z_k^\pm\varphi_k^\pm\in Z$ we define

$$\|z\|_s^2=\sum_{j=1}^n|z_j^\pm|^2 j^{2s},\quad s\in\mathbf{R} \tag{2.1}$$
$$y(z)=y_1^+\varphi_{n+1}^+ + y_1^-\varphi_{n+1}^- + y_2^+\varphi_{n+2}^+ +\ldots,\quad y_k^\pm=z_{n+k}^\pm$$

and define polar coordinates ξ_l, q_l in the planes (z_l^+, z_l^-), $l=1,\ldots,n$:

$$\xi_l=(z_l^{+2}+z_l^{-2})/2-I_l,\quad q_l=\operatorname{Arg}(z_l^+ + iz_l^-),\quad l=1,\ldots,n$$

In the neighbourhood of the torus $T^n(I)$ in Z we define coordinates (q, ξ, y), where

$$q\in T^n=\mathbf{R}^n/(2\pi\mathbf{Z}^n),\quad \xi\in O_\delta(\mathbf{R}^n),\quad y\in O_\delta(Y),\quad \delta>0$$

Eq.(1.7) has a very simple structure in terms of the coordinates (q, ξ, y): $q_k=\nu_k+$

$t\lambda_k$, $\xi = \text{const}$, $y = 0$. Let Z_s be the space of elements $z = \sum z_k^\pm \varphi_k^\pm$ with finite norm $\|z\|_s$ (see (2.1)), and $Y_s = Y \cap Z_s$ (in particular, $Z_0 = Z$, $Y_0 = Y$). By condition B, the operators J and A define continuous mappings $J: Z_{t+d_J} \to Z_t$, $A: Z_{t+d_A} \to Z_t$ for any $t \in \mathbf{R}$.

Definition. Let $M \geqslant 1$. A vector $\eta \in \mathbf{R}^n$ is said to be M-non-resonant if there **exists** $\rho > 0$ such that

$$|\eta \cdot s| \geqslant \rho |s|^{-n}, \quad \forall s \in \mathbf{Z}^n, \quad s \neq 0 \tag{2.2}$$

$$|\eta \cdot s \pm \lambda_k| \geqslant \rho (1 + |s|)^{-M}, \quad \forall k \geqslant n+1, \quad \forall s \in \mathbf{Z}^n \tag{2.3}$$

Otherwise η is called an M-resonant vector.

Proposition 1. If $d_1 > 0$, $\lambda_k \neq 0$ for all $k \geqslant n+1$ and $M > n + d_1^{-1} - 1$, then the set of all M-resonant vectors has measure zero.

Proof. It will suffice to prove that for $\rho < \min(1, \inf\{|\lambda_k|\})$ and any $L > 0$ the set of points $\eta \in O_L(\mathbf{R}^n)$ not satisfying condition (2.2) or (2.3) has measure at most $C\rho$, $C = C(L)$.

The set of points $\eta \in O_L(\mathbf{R}^n)$ violating condition (2.3) is the union of the layers

$$\Pi^{\pm}_{k,s} = \{\omega \in O_L(\mathbf{R}^n) \mid |\omega \cdot s \pm \lambda_k| < \rho(1 + |s|)^{-M}\}, \quad s \neq 0$$

If $|s| < \sigma_k$, where $\sigma_k = \max(1, (\lambda_k - 1)/L)$, then $\Pi^{\pm}_{k,s} = \varnothing$. But if $|s| \geqslant \sigma_k$, then the measure of the layer is at most $C_1 \rho |s|^{-M-1}$. Therefore

$$\operatorname{mes} \bigcup_s \Pi^{\pm}_{k,s} \leqslant C_1 \rho \sum_{|s| \geqslant \sigma_k} |s|^{-M-1} \leqslant C_2 \rho \sigma_k^{-M+n-1}$$

and by condition B the measure of the union of all non-empty layers is at most

$$C_3 \rho \Big(1 + \sum_{k \geqslant n+1} k^{-d_1(M-n+1)}\Big) \leqslant C_4 \rho$$

A similar estimate holds for the measure of the set of points $\eta \in O_L(\mathbf{R}^n)$ violating condition (2.2).

Let Z_d^C be a complexification of the space Z_d $(d \in \mathbf{R})$.

Theorem 1. Let conditions A and B be satisfied; assume that $d_1 = d_A + d_J > 0$, $\lambda_k \neq 0$ for $k \geqslant n+1$ and

1) there exists $d_0 > 0$ such that $H_\Delta(\cdot, \varepsilon)$ and $\nabla H_\Delta(\cdot, \varepsilon)$ can be continued as analytic mappings

$$H_\Delta(\cdot, \varepsilon): Z_{d_0}^C \to \mathbf{C}, \quad \nabla H_\Delta(\cdot, \varepsilon): Z_{d_0}^C \to Z^C_{d_0+d_J} \tag{2.4}$$

bounded on bounded sets uniformly in $\varepsilon \in (0, 1]$;

2) the vector $\omega^0 = (\lambda_1, \ldots, \lambda_n)$ is M-resonant for some $M \geqslant 1$.

Let $\rho > 0$ be the number corresponding to ω^0 figuring in the definition of M-non-resonance. Then constants K_1° and K_2 exists, independent of ρ and ε, such that if $z(0) = z_0 = (q_0, \xi_0, y_0)$ in (1.3) and for some a, $0 < a \leqslant 1$,

$$|\xi_0| + \|y_0\|_{d_0} \leqslant \varepsilon^a \rho^{-2} \tag{2.5}$$

then for sufficiently small $\varepsilon > 0$ the solution of Eq.(1.3) is such that $z(t)$ and $z^\cdot(t)$ are bounded in Z_{d_0} and $Z_{d_0 - d_J}$, respectively; it exists and is unique for $0 \leqslant t \leqslant L(\varepsilon) = \varepsilon^{-1} \ln \varepsilon^{-1}/K_1$, $K_1 \geqslant K_1^\circ$. The solution moreover satisfies the estimate

$$\|z(t) - (q_0 + t\omega^1, 0, 0)\|_{d_0} \leqslant K_2 \varepsilon^{a-\varkappa} \rho^{-2}, \; 0 \leqslant t \leqslant L(\varepsilon) \tag{2.6}$$

where $\omega^1 \in \mathbf{R}^n$ is the vector with components

$$\omega_j^1 = \lambda_j + \varepsilon \lambda_{jJ} \int_{T^n} \frac{\partial}{\partial \xi_j} H_\Delta(q, 0, 0; \varepsilon)\, dq = \int_{T^n} [q_j, H_0(\cdot; \varepsilon)](q, 0, 0)\, dq \tag{2.7}$$

and $\varkappa = \varkappa(K_1) > 0$, $\varkappa \to 0$ as $K_1 \to \infty$.

Remarks. 1°. If $M > n + d_1^{-1} - 1$, then it follows from Proposition 1 that the set of vectors ω° not satisfying condition 2 has measure zero.

2°. The second half of condition 1 is understood in the sense that estimate (4.2) (see below, Sect.4) is valid uniformly in $\varepsilon \in (0, 1]$.

3°. Higher-order averages have been constructed only for a few spatially one-dimensional systems of type (1.3) /7, 8/. For such systems $\lambda_j = K_0 j^{d_1} + o(j^{d_1-1}), d_1 \geqslant 1$. We showed /13/ that in that case the "non-resonant" conditionally periodic solutions of system (1.4) correspond to nearly conditionally periodic solutions of system (1.3).

3. *Applications to von Kármán's equations.* For simplicity, we will confine our attention of solutions of system (1.8) which are periodic functions of x:

$$u_j(t, x_1 + 2\pi, x_2) \equiv u_j(t, x_1, x_2 + 2\pi) \equiv u_j(t, x),\ j = 1, 2 \tag{3.1}$$

$$\int_0^{2\pi}\int_0^{2\pi} u_j(t, x_1, x_2)\, dx_1\, dx_2 \equiv 0, \quad j = 1, 2 \tag{3.2}$$

By (3.1), the functions $u_1(t, \cdot), u_2(t, \cdot)$ may be considered defined on a two-dimensional torus $T_x^2 = \mathbf{R}_x^2/2\pi\mathbf{Z}^2$. Let G_2 denote "Green's operator", i.e., the operator inverse to Δ^2 on T_x^2 under conditions (3.2). It then follows from (1.8) that

$$u_1^{\cdot\cdot} + a_1\Delta^2 u_1 + \varepsilon a_2^{-1}[u_1, G_2([u_1, u_1]')]' = 0 \tag{3.3}$$

Let L_0^2 denote the space of functions in $L^2(T_x^2)$ with zero average. The operator Δ^2 defines a selfadjoint positive operator N_2 in L_0^2 with domain $H_0^4(T_x^2)$ and a linear isomorphism $N_2 : H_0^2(T_x^2) \to H_0^{-2}(T_x^2)$ (where $H_0^j(T_x^2)$ is the subspace of $H^j(T_x^2)$ consisting of the functions with zero average; H^j are the Sobolev spaces). Put $N_1 = \sqrt{N_2}$, $G_1 = N_1^{-1}$; then $G_2 = G_1^2$.

Put $Z^\circ = H_0^2(T_x^2), \| u \|^2 = \iint (N_1 u)^2\, dx;\ Z = Z^\circ \times Z^\circ, \| (u, v) \|_0^2 = \| u \|^2 + \| v \|^2$. We define in Z an antiselfadjoint operator J and a functional H_Δ:

$$J(u, v) = (N_1 v, -N_1 u) \tag{3.4}$$

$$H_\Delta(u, v) = \iint (G_1([u, u]'))^2\, dx \tag{3.5}$$

Lemma 1. The functional H_Δ is analytic in Z and

$$\nabla H_\Delta(u, v) = 4(G_2[u, G_2[u, u]']', 0) \tag{3.6}$$

Proof. Given $u \in Z^\circ$, let $H_{\Delta 0}(u)$ equal the right-hand side of (3.5). To prove the lemma it will suffice to show that $H_{\Delta 0}$ is analytic on Z° and to determine its gradient. Analyticity follows from the estimate $H_{\Delta 0}(u) | \leqslant C \| u \|^4$ (see /12, 13/). For $u, v \in Z^0$ we have

$$dH_{\Delta 0}(u)v = 4\iint G_1([u, u]')\, G_1([u, v]')\, dx = 4\iint G_2([u, u]')\, [u, v]'\, dx$$

We know /12, 13/ that the trilinear form

$$(u, v, w) \mapsto \iint [u, v]'(x)\, w(x)\, dx$$

is symmetric. Therefore,

$$dH_{\Delta 0}(u)v = \iint v(x)\, [u, W]'(x)\, dx = \langle v, G_2[u, W]' \rangle$$

where $W = 4G_2[u, u]'$. Thus $\nabla H_{\Delta 0}(u) = G_2[u, W]'$, implying (3.6).

Consider the Hamiltonian

$$H_0(z) = {}^1/_2 \| z \|_0^2 \sqrt{a_1} + {}^1/_4 \varepsilon (a_2\sqrt{a_1})^{-1} H_\Delta(z),\ z = (u, v) \tag{3.7}$$

It has the form of the Hamiltonian of system (1.3) if A is taken to be the operator $\sqrt{a_1} I$ (I is the identity operator in Z). By Lemma 1,

$$\nabla H_0(u, v) = (\sqrt{a_1} u + \varepsilon (a_2\sqrt{a_1})^{-1} G_2[u, G_2[u, u]']', \sqrt{a_1} v)$$

Thus the system corresponding to H_0 is

$$u^\cdot = \sqrt{a_1} A_1 v,\ v^\cdot = -A_1(\sqrt{a_1} u + \varepsilon a_2^{-1} G_2[u, G_2[u, u]']' \tag{3.8}$$

If (u, v) is a solution of system (3.8), then $u(t, x)$ satisfies Eq.(3.3). Thus system (1.8) is equivalent to Eq.(1.3) with Hamiltonian (3.7), provided that the operator figuring in the definition of the Poisson bracket (1.1) is (3.4). Let $\{\psi_j \mid j \geqslant 1\}$ be a complete system of eigenfunctions of the operator N_2 and $N_2\psi_j = \mu_j^2\psi_j$. By the known asymptotic behaviour of the spectrum of an elliptic differential operator, $\mu_j = K_* j + o(j)$. Therefore, if $\varphi_j^+ = (\psi_j, 0)$, $\varphi_j^- = (0, \psi_j)$, then $J\varphi_j^\pm = \mp \mu_j \varphi_j^\mp$, $j = 1, 2, \ldots$, and the operators A and J corresponding to system (3.8) satisfy conditions A and B, with $\lambda_{jJ} = \mu_j$, $\lambda_{jA} = \sqrt{a_1}$, $d_A = 0$, $d_J = d_1 = 1$.

Lemma 2. For natural numbers s, the norm in Z_s is equivalent to the norm in $H_0^{2s+2}(T_x^2) \times H_0^{2s+2}(T_x^2)$.

Lemma 3. The mapping $\nabla H_\Delta\colon Z_m \to Z_{m+2}$ is analytic if $m \geqslant 2$.

The proof of Lemma 2 follows from the inequalities

$$C^{-1}\|(u,v)\|_s^2 \leqslant \|J^s(u,v)\|_0^2 \leqslant C\|(u,v)\|_s^2$$

since

$$\|N_1^s u\|^2 = \iint |N_1^{s+1}u|^2\,dx = \iint |\Delta^{s+1}u|^2\,dx$$

which is equivalent to the square of the norm in $H_0^{2s+2}(T_x^2)$.

To prove Lemma 3, we let $|\cdot|_s$ denote the norm in $H_0^s(T_x^2)$. Then $|G_2(u)|_{s+4} \leqslant C|u|_s$ for all s. If $s \geqslant 2$, then $|[u,v]'|_s \leqslant C_1|u|_{s+2}|v|_{s+2}$, whence it follows that the mapping $\nabla H_{\Delta 0}\colon H_0^m(T_x^2) \to H_0^{m+2}(T_x^2)$ is analytic for $m \geqslant 4$. Together with Lemma 2, this implies the required assertion.

Thus, system (3.8) satisfies the assumptions of Theorem 1 for $d_0 \geqslant 2$ and if the vector $\omega^0 = (\mu_1, \ldots, \mu_n)$ is non-resonant then the averaged solutions of system (3.8) are curves of type (1.7) with $\lambda_k = \omega_k^1$. The averaged solutions of Eq. (3.3) are the curves

$$u(t) = \sum_{k=1}^n \sqrt{2I_k}\sin(\omega_k^1 t + \nu_k)\psi_k(x)$$

4. *Proof of Theorem 1.* To abbreviate the discussion we omit the dependence of the Hamiltonian on ε; all estimates in this section are uniform in $\varepsilon \in (0, 1]$.

Transform system (1.3) from the variable z to variables $(q, \xi, y) \in O_{s,\delta} = T^n \times O_\delta(\mathbf{R}^n) \times O_\delta(Y_s)$, where

$$q_j^\cdot = \lambda_{jJ}\left(\lambda_{jA} + \varepsilon\frac{\partial}{\partial\xi_j}H_\Delta(q,\xi,y)\right),\quad \xi_j^\cdot = -\varepsilon\lambda_{jJ}\frac{\partial}{\partial q_j}H_\Delta(q,\xi,y)$$
$$y^\cdot = J(Ay + \varepsilon\nabla_y H_\Delta(q,\xi,y)) \tag{4.1}$$

The tangent space to $O_{s,\delta}$ at an arbitrary point is identified with Z_s.

Isolate the terms in H_Δ that are linear in ξ and y:

$$H_\Delta = g(q) + \xi\cdot h_1(q) + \langle y, \eta(q)\rangle + H_2(q,\xi,y)$$

Modifying H_Δ by a constant if necessary, we may assume that $\bar g = 0$ (the bar denotes averaging over $q \in T^n$). Write the Hamiltonian of system (4.1) as follows:

$$H = (\lambda_A{}^n + \varepsilon\bar h_1)\cdot\xi + {}^1\!/_2\langle Ay, y\rangle + \varepsilon H_1$$
$$\lambda_A{}^n = (\lambda_{1A}, \ldots, \lambda_{nA}),\quad H_1 = H_2 + H_3$$
$$H_3 = g(q) + \xi\cdot h(q) + \langle y, \eta(q)\rangle,\quad h = h_1 - \bar h_1$$

Continue the functional H_Δ analytically to a complex domain of the form

$$O^C_{d_0,\delta_1} = U_{\delta_1} \times O_{\delta_1}(\mathbf{C}^n) \times O_{\delta_1}(Y_{d_0}{}^C),\quad \delta_1 > 0$$
$$U_{\delta_1} = \{q \in \mathbf{C}^n/2\pi\mathbf{Z}^n \mid |\operatorname{Im} q| < \delta_1\}$$

where Y_s^C is a complexification of Y_s. By condition 1 of Theorem 1, the following estimate is true everywhere in $O^C_{d_0,\delta_1}$:

$$|H_\Delta(q,\xi,y)| + \|\nabla_y H_\Delta(q,\xi,y)\|_{d_0+d_J} \leqslant C \tag{4.2}$$

Our assertion now follows from (4.2) and the Cauchy inequality.

Lemma 4. There exist $\delta_2 > 0$ and constants C_1, C_2, C_3 such that for $q \in U_{\delta_2}$ and $(q, \xi, y) \in O^C_{d_0,\delta_2}$

$$|g(q)| + |h(q)| + |\bar h_1| + \|\eta(q)\|_{d_0+d_J} \leqslant C_1$$
$$|H_2(q,\xi,y)| \leqslant C_2(|\xi|^2 + \|y\|_{d_0}^2) \tag{4.3}$$
$$|\nabla_\xi H_2(q,\xi,y)| + \|\nabla_y H_2(q,\xi,y)\|_{d_0+d_J} \leqslant C_3(|\xi| + \|y\|_{d_0})$$

Define an auxiliary Hamiltonian $\varepsilon\Xi$, where

$$\Xi = g_0(q) + \xi\cdot h_0(q) + \langle y, \eta_0(q)\rangle$$

The corresponding canonical transformation S is the displacement per unit time along the trajectories of the Hamiltonian system with Hamiltonian $\varepsilon\Xi$:

$$q_j^{\cdot} = \varepsilon F_j^q(q), \quad \xi_j^{\cdot} = \varepsilon F_j^{\xi}(q, \xi, y), \quad y^{\cdot} = \varepsilon F^y(q) \tag{4.4}$$

$$F_j^q = \lambda_{jJ} h_{0j}(q), \quad F^y = J\eta_0(q)$$

$$F_j^{\xi} = -\lambda_{jJ}\left(\xi \cdot \frac{\partial h_0(q)}{\partial q_j} + \frac{\partial g_0(q)}{\partial q_j} + \left\langle y, \frac{\partial \eta_0(q)}{\partial q_j}\right\rangle\right) \tag{4.5}$$

S is a canonical transformation, taking system (4.1) into the system with Hamiltonian $H^1(q, \xi, y) = H(S(q, \xi, y))$ (see /3, 4/). Write S as follows: $q \mapsto q + \varepsilon q^1$, $\xi \mapsto \xi + \varepsilon\xi^1$, $y \mapsto y + \varepsilon y^1$. Then $q^1 = F^q + \varepsilon\ldots$, $\xi^1 = F^{\xi} + \varepsilon\ldots$, $y^1 = F^y + \varepsilon\ldots$. Therefore, putting $(q, \xi, y) = z$, $(q^1, \xi^1, y^1) = z^1$ and $\lambda_A{}^n + \varepsilon\bar{h}_1 = \omega^2$, we can write the transformed Hamiltonian as

$$H^1(z) = \omega^2 \cdot \xi + \frac{1}{2}\langle Ay, y\rangle + \varepsilon\Big(-\Big(\sum_{j=1}^{n} \xi \cdot \Big(\frac{\partial h_0}{\partial q_j}\lambda_{jJ}\omega_j^2\Big) + \frac{\partial g_0}{\partial q_j}\lambda_{jJ}\omega_j^2 + \Big\langle y, \frac{\partial\eta_0}{\partial q_j}\lambda_{jJ}\omega_j^2\Big\rangle\Big) + \langle Ay, J\eta_0(q)\rangle + g(q) + \xi \cdot h(q) + \langle y, \eta(q)\rangle\Big) + \varepsilon H_2(z) + O(\varepsilon^2)$$

Let $\omega^1 \in \mathbf{R}^n$ denote the vector with components $\omega_j^1 = \omega_j^2\lambda_{jJ}$. Since $h_{1j}(q) = \partial H_\Delta(q, 0, 0)/\partial\xi_j$ and for any functional H' we have $[q_j, H'] = \lambda_{jJ}\partial H'/\partial\xi_j$, it follows that $\lambda_{jJ}h_{1j}(q) = [q_j, H_0](q, 0, 0)$. Hence the vector ω^1 is of the form (2.7).

Equating the expression in square brackets to zero, we obtain homological equations for g_0, h_0 and η_0:

$$\partial g_0(q)/\partial\omega^1 \equiv \omega^1 \cdot \nabla g_0(q) = g(q) + \varepsilon\Delta g(q), \quad \partial h_0(q)/\partial\omega^1 = h(q) + \varepsilon\Delta h(q) \tag{4.6}$$

$$\partial\eta_0(q)/\partial\omega^1 - AJ\eta_0(q) = \eta(q) + \varepsilon\Delta\eta(q) \tag{4.7}$$

where $\varepsilon\Delta g$, $\varepsilon\Delta h$ and $\varepsilon\Delta\eta$ are admissible small increments.

Lemma 5. For some $\delta_3 > 0$,

a) there exist functions g_0, Δg, h_0 and Δh, analytic in U_{δ_3} and satisfying (4.6), such that everywhere in U_{δ_3}

$$|\Delta g(q)| + |\Delta h(q)| \leqslant C\varepsilon, \; |g_0(q)| + |h_0(q)| \leqslant C\rho^{-1} \tag{4.8}$$

b) There exist mappings η_0, $\Delta\eta$, analytic in U_{δ_3} and satisfying (3.7), such that everywhere in U_{δ_3}

$$\|\eta_0(q)\|_{d_0+d_J+d_1} \leqslant C\rho^{-1}, \quad \|\Delta\eta\|_{d_0+d_J} \leqslant C\varepsilon \tag{4.9}$$

Proof. We will confine ourselves to proving the more difficult assertion (b). To that end we define $W_j^{\pm} = (\varphi_j^+ \pm i\varphi_j^-)/\sqrt{2}$. Then $AJW_j^{\pm} = \pm i\lambda_j W_j^{\pm}$. Expand the mappings η_0, η and $\Delta\eta$ in terms of the basis elements $W_j^{\pm}: \eta(q) = \sum \eta_k^{\pm}(q) W_k^{\pm}$ etc. Functions of s will denote Fourier transforms with respect to q:

$$\eta_k^{\pm}(q) = \sum_{s \in Z^n} \eta_k^{\pm}(s) e^{i}$$

and so on. It follows from Lemma 4 and known estimates for the decrease of the Fourier coefficients of analytic functions (/14/, Sect.4.2) that $\|\eta(s)\|_{d_0+d_J} \leqslant C\exp(-\delta_2|s|)$. Hence there exists a number $C_* = C_*(\delta_2 - \delta_3)$ such that if

$$\Delta\eta = \sum_{|s|>M_*} \eta(s) e^{iq\cdot s}, \quad M_* = C_* \ln \varepsilon^{-1} \tag{4.10}$$

then $\Delta\eta$ satisfies estimate (4.9). Then the quantities $\eta_{0k}^{\pm}(s)$ vanish for $|s| > M_*$, while for $|s| \leqslant M_*$

$$\eta_{0k}^{\pm}(s) = -i\eta_k^{\pm}(s)/(\omega^1 \cdot s \mp \lambda_k) \tag{4.11}$$

Since

$$|\omega^1 \cdot s \mp \lambda_k| \geqslant |\omega^0 \cdot s \mp \lambda_k| - \varepsilon|\omega^3 \cdot s|, \quad \omega_j^3 = \bar{h}_{1j}\lambda_{jJ}$$

it follows from condition 2 of the theorem that

$$|\omega^1 \cdot s \mp \lambda_k| \geqslant {}^1/_2\rho(1 + |s|)^{-M} + {}^1/_2\rho(1 + C_* \ln \varepsilon^{-1})^{-M} - C_1\varepsilon \ln \varepsilon^{-1}$$

Therefore, if ε is sufficiently small, the absolute value of the denominator in (4.11) is at least $\frac{1}{2}\rho(1+|s|)^{-M}$ and

$$\|\eta_0(q)\|_{d_0+d_J}+\|\partial\eta_0(q)/\partial\omega^1\|_{d_0+d_J}\leqslant C_2\rho^{-1},\quad q\in U_{\delta_3},\ \delta_3<\delta_2$$

(see /14/). Consequently, it follows from the estimate for $\Delta\eta$ and the form of Eq.(4.7) that $\|AJ\eta_0(q)\|_{d_0+d_J}\leqslant C_3\rho^{-1}$ for $q\in U_{\delta_3}$. The estimate for $\eta_0(q)$ now follows from condition B.

As bfore, let S be the displacement operator along trajectories of (4.4) and let $S(z)=z+\varepsilon z^1$. Then if $z(t)$ is the trajectory of (4.4) with initial condition z, we have

$$z^1=\int_0^1 F(z(t))\,dt,\quad F=(F^q,F^\xi,F^y)$$

Thus Lemma 5 implies the following

Lemma 6. There exists $\delta_4>0$ such that the mapping $S\colon O_{s,\delta_4}\to O_{s,\delta_4}$ is analytic for $s\in[-s_1,\ s_2]$, $s_1=d_0+d_J+d_1$, $s_2=d_0+d_J+d_A$. Moreover,

$$|q^1|\leqslant C\rho^{-1},\quad |\xi^1|\leqslant C\rho^{-1}(1+\|y\|_{-s_1}),\quad \|y^1\|_{s_2}\leqslant C\rho^{-1}$$
$$\forall s\in[-s_1,s_2],\quad \|dS(z)\|_{Z_s,Z_s}\leqslant 2,\quad \|dS(z)-I-\varepsilon dF(z)\|_{Z_s,Z_s}\leqslant C\varepsilon^2\rho^{-2}$$

(we recall that ε is assumed to be sufficiently small and the tangent space to $O_{s,\delta}$ is identified with Z_s).

Let $z\in O_{s,\delta_4}$, $S(z)=z+\varepsilon z^1$. Then

$$\begin{gathered}H(S(z))=\omega^2\cdot\xi+\tfrac{1}{2}\langle Ay,y\rangle+\varepsilon[(\xi^1-F^\xi)\cdot\omega^2]_1+\\ \varepsilon[\langle Ay,y^1-F^y\rangle]_2+\tfrac{1}{2}\varepsilon^2[\langle Ay^1,y^1\rangle]_3+\varepsilon[-\partial g_0/\partial\omega^1+g]_4+\\ \varepsilon[(-\partial h_0/\partial\omega^1+h)\cdot\xi]_5+\varepsilon[\langle\partial\eta_0/\partial\omega^1-AJ\eta_0-\eta,y\rangle]_6+\\ \varepsilon[H_1(z+\varepsilon z^1)-H_1(z)]_7+\varepsilon H_2(z)\end{gathered}$$

Denote the functional in square brackets $[\cdot]_j$ (together with its coefficient) by Δ_jH. Lemmas 5 and 6 imply the following.

Lemma 7. If $z\in O_{d_0,\delta_5}$, where $\delta_5<\delta_4$, then for $j=1,\ldots,7$

$$|\Delta_jH|+\|\nabla_y\Delta_jH\|_{d_0+d_J}\leqslant C\varepsilon^2\rho^{-2}$$

Thus,

$$H(S(z))=\omega^2\cdot\xi+\tfrac{1}{2}\langle Ay,y\rangle+\varepsilon^2H_4(z)+\varepsilon H_2(z)$$

where the functionals $H_2(z)$ and $H_4(z)$ are analytic in O_{d_0,δ_5} and

$$|H_4(z)|+\|\nabla_yH_4(z)\|_{d_0+d_J}\leqslant C\rho^{-2}\tag{4.12}$$

The transformed system of equations may be written as

$$q_j^{\cdot}=\omega_j^1+\varepsilon\lambda_{jJ}(\partial/\partial\xi_j)(\varepsilon H_4+H_2)\tag{4.13}$$
$$\xi_j^{\cdot}=-\varepsilon\lambda_{jJ}(\partial/\partial q_j)(\varepsilon H_4+H_2)\tag{4.14}$$
$$y^{\cdot}=J(Ay+\varepsilon^2\nabla_yH_4+\varepsilon\nabla_yH_2)\tag{4.15}$$

Since the mappings $J\nabla_yH_4$ and $J\nabla_yH_2$ are analytic (see (4.12) and Lemma 4), this system is obtained by perturbing the system $q^{\cdot}=\omega^1$, $\xi^{\cdot}=0$, $y^{\cdot}=JA_y$ by a vector field satisfying a Lipschitz condition. Since the unperturbed system defines a group of isometric transformations of the domain O_{d_0,δ_6}, $\delta_6<\delta_5$, the solution of system (4.13)-(4.15) is unique and exists at least up to the time at which the boundary of the domain O_{d_0,δ_5} is reached (see, e.g., /15/, p.105). Let $(q(t),\ \xi(t),\ y(t))=z'(t)$ be the solution of the system such that $S(z'(0))=z_0=(q_0,\ \xi_0,\ y_0)$. By (4.14), Lemma 4, estimate (4.12) and the Cauchy inequality, we have

$$d|\xi(t)|/dt\leqslant\varepsilon C(\varepsilon\rho^{-2}+|\xi|^2+\|y\|_{d_0}^2)\tag{4.16}$$

Let P be the linear operator carrying $\varphi_j^{\pm}$ into $j^{d_0}\varphi_j^{\pm}$, $j=1,2,\ldots$. Then

$$\|Py\|_0^2=\langle P^2y,y\rangle=\|y\|_{d_0}^2,\ \langle JAy,Py\rangle\equiv 0\tag{4.17}$$

Multiplying Eq.(4.15) by $P^2y(t)$ in Y (scalar product) and using (4.17), we obtain

$$^1/_2 d\langle y, P^2y\rangle/dt = \varepsilon^2\langle J\nabla_y H_4, P^2y\rangle + \varepsilon\langle J\nabla_y H_2, P^2y\rangle$$

By (4.12) and Lemma 4, the right-hand side of this equality does not exceed: $C\varepsilon\|y\|_{d_0}$ $(\varepsilon\rho^{-2} + |\xi| + \|y\|_{d_0})$. Hence

$$d\|y\|_{d_0}/dt \leqslant C_1(\varepsilon^2\rho^{-2} + \varepsilon|\xi| + \varepsilon\|y\|_{d_0}) \tag{4.18}$$

Assume that $\|y(t)\|_{d_0} + |\xi(t)| \leqslant 1$. Then by (4.16), (4.18),

$$(d/dt)(|\xi(t)| + \|y(t)\|_{d_0}) \leqslant C_2\varepsilon(|\xi| + \|y\|_{d_0} + \varepsilon\rho^{-2})$$

Hence, by Gronwall's Lemma,

$$|\xi(t) + \|y(t)\|_{d_0} \leqslant (\varepsilon\rho^{-2} + |\xi(0)| + \|y(0)\|_{d_0})\, e^{C_2\varepsilon t} - \varepsilon\rho^{-2} \tag{4.19}$$

By condition (2.5) and Lemma 6, $|\xi(0)| + \|y(0)\|_{d_0} \leqslant C_3\rho^{-2}\varepsilon^a$. Therefore, if $0 < 2b < a \leqslant 1$, then for $0 \leqslant t \leqslant L(\varepsilon) = b\ln\varepsilon^{-1}/(\varepsilon C_2)$ we have

$$|\xi(t)| + \|y(t)\|_{d_0} \leqslant C_4(\varepsilon^a\rho^{-2})\varepsilon^{-b} \tag{4.20}$$

Consequently, if ε is sufficiently small, the solution $z'(t)$ exists at least for $0 \leqslant t \leqslant L(\varepsilon)$. If $z_*(t) = (q_0 + \omega^1 t, 0, 0)$, it follows from (4.20) and (4.13) that

$$\|z'(t) - z_*(t)\|_{d_0} \leqslant C_5\varepsilon^{a-b}\rho^{-2}$$

Therefore, by the estimates in Lemma 6,

$$\|S(z'(t)) - z_*(t)\|_{d_0} \leqslant \|S(z'(t)) - S(z_*(t))\|_{d_0} + \|S(z_*(t)) - z_*(t)\|_{d_0} \leqslant$$
$$2\|z'(t) - z_*(t)\|_{d_0} + C_6\varepsilon\rho^{-1} \leqslant C_7\varepsilon^{a-b}\rho^{-2}$$

But $S(z'(t)) = z(t)$ is the solution of system (1.3) with initial condition z_0. Estimate (2.6) is proved if one puts $\varkappa = 2b$ (and if ε is sufficiently small).

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